

# Emergent Spacetime and Black Hole Probes from Automorphic Forms

ROLF SCHIMMRIGK<sup>1,2</sup>

Theory Division, CERN  
1211 Geneva 23, Switzerland

## Abstract

Over the past few years the arithmetic Langlands program has found applications in two quite different problems that arise in string physics. The first of these is concerned with the fundamental problem of deriving the geometry of spacetime from the worldsheet dynamics, leading to a realization of the notion of an emergent spacetime in string theory. The second problem is concerned with the idea of using automorphic black holes as probes of spacetime. In this article both of these applications of the Langlands program are described.

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<sup>1</sup>On leave from Indiana University South Bend

<sup>2</sup>netahu@yahoo.com, rschimmr@iusb.edu

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Emergent spacetime from string modular forms</b>	<b>3</b>
2.1	Emergent space from worldsheet modular forms . . . . .	3
2.2	Construction of motives . . . . .	5
2.3	A speculative framework for a proof . . . . .	6
2.4	An elliptic example . . . . .	7
2.5	Higher dimensional diagonal varieties and their families . . . . .	8
2.6	Arithmetic mirror symmetry . . . . .	9
<b>3</b>	<b>Automorphic black holes as probes of extra dimensions</b>	<b>10</b>
3.1	Automorphic black holes . . . . .	10
3.2	Siegel modular black holes in $\mathcal{N} = 4$ theories . . . . .	12
3.3	Automorphic motives . . . . .	13
3.4	Lifts of weight two forms . . . . .	14

## 1 Introduction

Applications of arithmetic algebraic geometry have been rare in particle physics in general and are not common in string theory. One reason for this might be that the methods involved are far removed from the tools contained in the standard kit issued to theoretical physicists. There is one part of number theory, however, that one might expect to be of relevance for string physics, and that is the theory of modular forms, in particular the link of modular forms to geometry. This is an old subject, with roots that can be traced back to the work of Klein and his student Hurwitz more than a century ago. This line of thought was continued, with interruptions, by Hecke, Eichler, Taniyama, Shimura, Weil, as well as Wiles and Taylor, in the context of elliptic curves. Its relevance for string physics, however, depends in an essential

way on the important extension of the elliptic setting by Grothendieck, Langlands, Deligne, Serre, and others, into what today is called the Langlands program.

The framework provided by the Langlands program consists of several difficult conjectures about automorphic forms, of which the most important for the applications discussed here is what is often called the reciprocity conjecture. It is concerned with the geometric origin of automorphic forms and their associated representations and in very rough terms states that all motives are automorphic. In a first approximation motives can be thought of as geometric building blocks, whose lego-like structure allows to build full-fledged manifolds. While the inverse of this conjecture is not believed to be true, at least within the current framework of both automorphic forms and motives, it is expected that a certain sub-class of automorphic forms is motivic.

The two ideas described in this review are both concerned with the geometric nature of automorphic forms that arise in string theory. The first addresses the old problem of deriving the geometry of spacetime from string physics. Given that string theory is expected to be a complete theory one might expect that it should be possible to derive not only a model for particle physics that extends the standard model, but also the arena where particle interactions takes place, i.e. the structure of spacetime itself. This vague expectation can be made precise within the framework of the Langlands program because the conformal field theory on the string worldsheet leads to modular forms that encode the fundamental physics of the string. One can therefore ask whether these modular forms can be used to construct geometric motives that provide the building blocks of the extra dimensions. By combining several of these motives one can hope to build the compactification manifold from scratch. The review of this problem is based mostly on the work of [1, 2], and references therein.

The second problem is concerned with a new application of black holes that can be suggested in light of the Langlands program. In this context the automorphic forms arise as functions that describe the entropy of black holes in certain classes of string compactifications with extended supersymmetry. Given such automorphic black hole entropy functions, one can ask whether they encode motivic information about the compact part of spacetime, leading to the view that automorphic black holes can be used as physical probes that are sensitive to the

nature of the extra dimensions. The modularity of the entropy implies that if one were to experiment with such black holes in the laboratory, a finite number of measurements would suffice to completely determine these functions. The review of this problem is based on [3].

## 2 Emergent spacetime from string modular forms

Speculations about the nature of spacetime have a long history which can be traced back at least to Born's 1919 pre-Heisenberg thinking that a true understanding of quantum theory will very likely necessitate a drastic change in the view of spacetime as a manifold over the real numbers [4]. In more recent times this has led to the speculation that spacetime might be  $p$ -adic in nature. A more natural view would be to combine all  $p$ -adic numbers into an adelic structure.

The view that at some fundamental level spacetime is adelic in nature clashes with the fact that in physics the dynamics is encoded in differential equations, which motivates the assumption that spacetime is defined over the field of real numbers  $\mathbb{R}$ , in order to be able to do analysis. In the case of higher dimensional theories with extra dimensions supersymmetry leads to compact spaces that are defined over the complex numbers  $\mathbb{C}$ . A more pragmatic point of view of number theoretic methods in this complex framework is to think of restrictions of spacetime to number fields, instead of  $\mathbb{R}$  or  $\mathbb{C}$ , as a particular type of lattice approximation, not necessarily as a radical reinterpretation of the nature of spacetime. It is this more pragmatic latter attitude that has led to some results in the understanding of how space could be constructed from the physics on the worldsheet. This does not preclude that adelic features might eventually emerge as fundamental properties, but such an assumption is not necessary for physical applications of number theoretic methods.

### 2.1 Emergent space from worldsheet modular forms

The basic idea of the emergent spacetime program initiated in [5, 6], and further developed in [7, 1], is to use modular forms that arise in the conformal field theoretic models on the world-

sheet  $\Sigma$  to derive the geometric structure of the extra compact dimensions in the spacetime manifold  $X$ . This program can be viewed as a string theoretic refinement of the Langlands program, part of which leads to the expectation that a particular class of automorphic forms are supported by geometric structures called motives, denoted here generically by  $M$  [8, 9, 10]. Motives in turn can be viewed as support structures for certain types of cohomology groups. While these motivic cohomology groups are not the ones that have traditionally been encountered in physics, there exist certain compatibility theorems which imply that their rank is identical to that of cohomology groups that are more familiar from string theory, such as the Hodge decomposition of the de Rham cohomology for complex varieties.

The number theoretic nature of these motives can not be avoided because the only known way to obtain automorphic forms from these objects  $M$  is via the computation of their  $L$ -function  $L(M, s)$ , a function in the complex variable  $s \in \mathbb{C}$  that a priori is only defined over part of the complex plane  $\mathbb{C}$ . It is, however, conjectured that this function can be continued to all  $s$ , and that it satisfies a functional equation. This functional equation encodes Poincaré duality of the motives, and is a first indication of the automorphy of the associated  $q$ -series.

The results obtained so far are mostly concerned with modular motives in the sense that the inverse Mellin transform  $f(M, s)$  of the motivic  $L$ -series  $L(M, s)$  leads to modular forms, in particular to cusp forms of weight  $w$  and level  $N$  with respect to the Hecke congruence subgroup

$$f(M, q) \in S_w(\Gamma_0(N)). \quad (1)$$

Here the weight  $w$  of the form is determined by the weight  $\text{wt}(M)$  of the motive  $M$  via

$$w(f) = \text{wt}(M) + 1, \quad (2)$$

where the weight  $\text{wt}(M)$  of the motive can be viewed as the degree of the associated cohomology group

$$\text{wt}(M) := \deg H(M) = r \quad (3)$$

for  $H(M) \subset H^r(X)$ . This relation assumes that the motive  $M$  is pure, i.e. it arises within a smooth variety. This is the situation that applies to the case of weighted Fermat hypersurfaces considered in [7, 1, 11]. The case of mixed motives, much less understood, applies in the

context of phase transitions between Calabi-Yau varieties and has been considered in [2] in the context of extending this program from the case of weighted Fermat spaces to families of manifolds.

## 2.2 Construction of motives

One of the problems encountered in the context of finding a string theoretic interpretation of motivic automorphic forms is that in general a Calabi-Yau variety of arbitrary dimension has several nontrivial cohomology groups  $H^r(X)$ . For each of these groups a decomposition into their irreducible motivic subgroups

$$H^r(X) = \bigoplus_i H(M_i) \quad (4)$$

does not appear to be known in mathematics. Given a variety, the first question thus becomes how one should construct motives. For the simplest Calabi-Yau spaces, i.e. elliptic curves  $E$ , this is clear because there is only one motive  $M_E$ , leading to the first cohomology group

$$H(M_E) = H^1(E). \quad (5)$$

The conjectures of Taniyama, Shimura, and in particular Weil made it possible to determine for any given elliptic curve  $E$  the modular form of the  $L$ -series  $L(M_E, s)$  by computing the conductor  $N$  of the elliptic curve  $E$  and comparing the result with the basis vectors in the finite dimensional space  $S_2(\Gamma_0(N))$ .

For higher dimensional Calabi-Yau manifolds (and more general spaces) a universal construction was given in [1], generalizing to all Calabi-Yau varieties the special case of weighted hypersurfaces described in [7]. This construction is based on the existence of the intermediate cohomology group, in particular the existence of the holomorphic  $(n, 0)$ -form, usually called  $\Omega \in H^{n,0}(X)$ , for a Calabi-Yau  $n$ -fold  $X$ . For this reason the motive derived from this form is called the  $\Omega$ -motive [7, 1]. The generalization of the  $\Omega$ -motive to a more general class of special Fano varieties [12, 13, 14] leads to a "twisted" cohomology group  $H^{n-w,w}(X)$  [11]. More precisely, the construction of this motive is based on a Galois group  $\text{Gal}(K_X/\mathbb{Q})$  of a number field  $K_X$  that is derived from the arithmetic structure of the manifold.

For modular  $\Omega$ -motives of Calabi-Yau manifolds the relation (2) between the weight of the modular form and the weight of the motive leads to a direct relation between the weight of the modular form  $f_\Omega$  associated to  $M_\Omega$  and the dimension of the variety as

$$w(f_\Omega) = \dim_{\mathbb{C}} X + 1. \quad (6)$$

For CY threefolds the relevant forms are therefore of weight four and some level  $N$ ,  $f_\Omega \in S_4(\Gamma_0(N))$ , where  $N$  is determined by the structure of the motive  $M_\Omega$  in a way that is not understood at present.

### 2.3 A speculative framework for a proof

It is not clear a priori what the correct framework should be in which one might attempt to establish a proof of the relation conjectured to exist between the modular forms on the worldsheet  $\Sigma$  on the one side, and modular, or automorphic, motives of spacetime on the other. It is nevertheless tempting to identify the structures that appear in the examples considered so far, and to speculate what might be some of the concepts that should be key ingredients of a proof. First, on the geometric side of the conjectured relation there should exist a set of automorphic forms that are associated to motives that arise in spaces that are either of Calabi-Yau type or of special Fano type. Denote the set of all motives associated to Calabi-Yau and special Fano type varieties by  $\mathcal{M}_{\text{CYF}}$ , and denote the set of all automorphic forms of all motives by  $\mathcal{M}_{\text{CYF}}$  by  $\mathcal{A}(\mathcal{M}_{\text{CYF}})$ . On the string theoretic side one might envision a second set  $\mathcal{A}(\text{CFT}_{\mathcal{N}=2})$  consisting of automorphic forms that arise from conformal field theories that live on the string worldsheet  $\Sigma$ . The hoped-for relation would then take the form of an identification of these two rather large sets

$$\mathcal{A}(\text{CFT}_{\mathcal{N}=2}) = \mathcal{A}(\mathcal{M}_{\text{CYF}}). \quad (7)$$

The above outline is too simplistic and has too little structure to be of ultimate use. It is natural to expect that a proof of the relation (7) is best approached in terms of structures that allow to build complicated objects from simpler ones, which means that one should endow the sets considered above with categorical notions. On the geometric side this is possible

at present for the case of smooth manifolds. In this case there exists a category  $\mathcal{M}$  of so-called pure motives. This category admits a tensor construction, hence it can be viewed as providing the building blocks of smooth projective manifolds. The existence of phase transition between Calabi-Yau manifolds [15], leading to the idea of a connected universal moduli space of string compactifications [16, 17], necessitates the consideration of the more general concept of mixed motives, as shown in [2]. The notion of a category of mixed motives has proven to be problematic and has not been accomplished yet in a satisfactory way. On the string theoretic side the idea would be to consider a category of conformal field theories with  $\mathcal{N} = 2$  supersymmetry, as well as their associated automorphic forms. At present the notion of a category of conformal field theories has also not been formalized.

## 2.4 An elliptic example

This subsection illustrates the construction of the extra dimensions from string worldsheet modular forms in the simplest possible example, that of a compactification of string theory to eight dimension on a two-torus. More details can be found in [6]. Suppose the worldsheet theory is given by the Gepner model based on the tensor product  $1^{\otimes 3}$  of three  $\mathcal{N} = 2$  superconformal minimal models, each with conformal level  $k = 1$  and central charge  $c(k) = 3k/(k+2) = 1$ . The first problem that arises is that in general such minimal models lead to many modular forms, and it is a priori not obvious which modular form one should consider. It turns out that the key structure that is of importance are the Hecke indefinite modular forms, given in terms of the parafermionic partition functions given by the Kac-Petersson string functions  $c_{\ell,m}^k(\tau)$  [18]. More precisely, the correct form to consider is the weight one forms

$$\Theta_{\ell,m}^k(\tau) = \eta^3(\tau) c_{\ell,m}^k(\tau), \quad (8)$$

where  $\eta(\tau)$  is the Dedekind eta function.

The example  $1^{\otimes 3}$  is a make-or-break case because for conformal level  $k = 1$  the model has only a single independent Hecke indefinite modular form  $\Theta_{1,1}^1(\tau)$ , hence there is no room to maneuver — this form either leads to the correct elliptic curve, or the whole set-up fails. Elliptic curves  $E_N$  of conductor  $N$  are known to be modular in the sense that the inverse



Mellin transform of their  $L$ -function are modular forms of weight two with respect to the Hecke congruence subgroup of level  $N$ , i.e.  $\Gamma_0(N) \subset \mathrm{SL}(2, \mathbb{Z})$  [19, 20, 21]. In concrete cases this can always be checked by a finite computation without the Galois machinery of Wiles et al. The general relation between the dimension of a Calabi-Yau variety and the weight of the modular form tells us that for elliptic curves the motivic form has to have weight two. This means that the motivic form should be a product  $\Theta_{1,1}^1(a\tau)\Theta_{1,1}^1(b\tau)$  for some integers  $a, b \in \mathbb{N}$ . These integers can be determined by a variety of criteria described in detail in [22], leading to a weight two form of level  $N = 27$

$$f_2(\tau) = \Theta_{1,1}^1(3\tau)\Theta_{1,1}^1(9\tau) \in S_2(\Gamma_0(27)). \quad (9)$$

From this modular form the elliptic curve of conductor 27 can be determined to be the cubic Fermat curve embedded in projective space  $\mathbb{P}_2$ , consistent with previous conjectures.

## 2.5 Higher dimensional diagonal varieties and their families

Higher dimensional extensions of the example above have been described both for the case of diagonal Calabi-Yau hypersurfaces [7, 1], diagonal weighted hypersurfaces of special Fano type [11], as well as for families of Calabi-Yau varieties [2]. The diagonal varieties considered in these references are related to Gepner models, constructed via tensor products of central charge  $c = 3D$ , where  $D = \dim_{\mathbb{C}} X$ , of  $\mathcal{N} = 2$  supersymmetric minimal models. This class of models [23] has been constructed explicitly, and their cohomological spectra have been computed in [24, 25].

Diagonal varieties are special because their underlying conformal field theory is rational. These spectra of these CFTs contain marginal operators, and it is of importance to consider deformations of the diagonal points. Conformal field theoretically, such deformations have proven difficult, and much less is known about them. In [2] a first exploration was initiated of modular motives that arise from singular Calabi-Yau varieties that appear in families of deformations of weighted hypersurfaces. This work uncovered some unexpected features of modular phases that involve non-conifold type singularities. First, depending on the degree of the singularity, degenerate  $\Omega$ -motives can be modular, even though naively they appear to be of high rank

(hence in the smooth case would lead to automorphic forms, rather than modular forms). Second, the modular forms that can be identified from the  $L$ -functions of these degenerate  $\Omega$ -motives can sometimes be shown to arise from modular forms that appear in smooth weighted Fermat hypersurfaces. This means that highly degenerate deformations of Gepner models can lead to the same modular forms that arise in the Gepner models themselves.

The phenomena observed in [2] can be interpreted as an indication that highly singular Calabi-Yau varieties that appear as intermediate phases in transitions between different Calabi-Yau manifolds define consistent string configurations.

## 2.6 Arithmetic mirror symmetry

The construction of the compact geometry directly from the worldsheet theory leads to a number of new questions in the context of mirror symmetry. The discovery of mirror symmetry in the context of weighted Calabi-Yau hypersurfaces [26] can be combined with the quotient mirror construction of Greene and Plesser [27] to formulate a mirror map directly at the level of Calabi-Yau varieties [28]. Since the quotient construction itself leads to an isomorphic conformal field theory it is clear that the mirror theory contains the same modular forms as the original theory, hence one might expect that certain arithmetic aspects of the mirror manifold are identical to that of the original theory. This notion is problematic because the cohomology groups of mirror pairs are completely different. While this problem has not been resolved (see [29]), there are some simple tests that can be performed. The simplest possible case is to consider elliptic compactifications, i.e. Gepner tensor models with total central charge  $c = 3$ , of which there are only three. The mirror construction of [28] leads to a mirror pairing, and one can ask whether these mirror pairs have the same  $L$ -function. The answer to this is affirmative [30], leading to a first arithmetic test of the geometric mirror construction of [28].

A key problem in mirror symmetry is to find the appropriate class of varieties that allows to construct mirrors of rigid Calabi-Yau manifolds. One class of varieties that has been suggested as such a framework is defined by a special class of Fano varieties [12, 13, 14]. These

varieties are of higher dimension than their Calabi-Yau counterparts, but contain motives that encode information about mirrors. It was shown in [11] that the  $L$ -series of the Fano varieties associated to two particular Gepner models are modular, and that the corresponding modular forms agree with the modular forms previously computed [31] for two rigid Calabi-Yau threefolds whose cohomology is mirror symmetric to that of the Fano varieties.

### 3 Automorphic black holes as probes of extra dimensions

The remainder of this review is dedicated to the description of a second recent program to apply ideas of the arithmetic Langlands program in physics. The problem posed here is what information could be extracted from black holes if we were able to experiment with them in the laboratory. More precisely, if the entropy of black holes is described by automorphic forms, as outlined below, then one might expect that such automorphic black holes encode information about the geometry of the extra dimensions in string theory. This raises the question of how one can identify the motives of the variety which support the automorphic forms that appear in the entropy results. This problem was addressed in [3], on which the discussion of this section is based.

#### 3.1 Automorphic black holes

Over the past 15 years impressive progress has been made toward the resolution of a problem that is almost 40 years old - the microscopic understanding of the entropy of black holes. It has been proven useful to focus on black holes with extended supersymmetries because this leads to black holes that are simple, but not too simple. It was shown in particular for certain types of black holes in  $\mathcal{N} = 4$  supersymmetric theories that their entropy is encoded in the Fourier coefficients of Siegel modular forms, automorphic forms that form one of the simplest generalizations of classical modular forms of one variable with respect to congruence subgroups of the full modular group  $\mathrm{SL}(2, \mathbb{Z})$ .

The general conceptual framework of automorphic entropy functions has not been formalized yet. A formulation that generalizes the existing examples can be outlined as follows. Suppose we have a theory which contains scalar fields parametrized by a homogeneous space  $\prod_i (G_i/H_i)$ , where the  $G_i$  are Lie groups. Associated to these scalar fields are electric and magnetic charge vectors  $Q = (Q_e, Q_m)$ , taking values in a lattice  $\Lambda$  whose rank is determined by the groups  $G_i$ .

Assume now that the theory in question has a T-duality group  $\prod_i D_i(\mathbb{Z})$ , where  $D_i(\mathbb{Z}) \subset G_i(\mathbb{Z})$  denotes the Lie groups considered over the rational integers  $\mathbb{Z}$ . Suppose further that the charge vector  $Q$  leads to norms  $\|Q\|_i$ ,  $i = 1, \dots, r$  that are invariant under the T-duality group. Choose conjugate to these invariant charge norms complex chemical potentials

$$(\tau_i, \|Q\|_i), \quad i = 1, \dots, r, \quad (10)$$

which generalize the upper half plane of the bosonic string. On this generalized upper half plane  $\mathcal{H}_r$  formed by the variables  $\tau_i$  one can consider automorphic forms  $\Phi(\tau_i)$ , and the idea is that with an appropriate integral structure  $\mathbb{Z} \ni k_i \sim \|Q\|_i, i = 1, \dots, r$  associated to the charge norms, the Fourier expansion of these automorphic forms given by

$$\Phi(\tau_i) = \sum_{k_n \in \mathbb{Z}} g(k_1, \dots, k_r) q_1^{k_1} \cdots q_r^{k_r}, \quad (11)$$

in terms of  $q_k = e^{2\pi i \tau_k}$ , determines the automorphic entropy via the coefficients of the expansion of the automorphic partition function

$$Z = \frac{1}{\widetilde{\Phi}} = \sum_{k_n} d(k_1, \dots, k_r) q_1^{k_1} \cdots q_r^{k_r}, \quad (12)$$

as

$$S_{\text{mic}}(Q) \sim \ln d(Q), \quad (13)$$

where

$$d(Q) := d(\|Q\|_1, \dots, \|Q\|_r). \quad (14)$$

Here  $\widetilde{\Phi}$  denotes a modification of the Siegel form  $\Phi$  that is determined by the divisor structure of  $\Phi$ .

### 3.2 Siegel modular black holes in $\mathcal{N} = 4$ theories

The above automorphic-entropy-outline describes the behavior of the entropy of black holes in certain  $\mathcal{N} = 4$  compactifications obtained by considering  $\mathbb{Z}_N$ -quotients of the heterotic toroidal compactification  $\text{Het}(T^6)$ , a small class of models first considered by Chaudhuri-Hockney-Lykken models [32]. Specifically, it was shown in [33, 34, 35] that for these  $\text{CHL}_N$  models the microscopic entropy of extreme Reissner-Nordstrom type black holes is described by Siegel modular forms  $\Phi^N \in S_w(\Gamma_0^{(2)}(N))$ , where the weight  $w$  is determined by the order  $N$  of the quotient group. In this case the dyonic charges  $Q = (Q_e, Q_m)$  form three integral norms invariant under the T-duality group  $\text{SO}(6, n_v)$ , where  $n_v$  depends on the group order  $N$ . These norms are usually denoted by

$$\|Q\|_1 = \frac{1}{2}Q_e \cdot Q_e, \quad \|Q\|_2 = \frac{1}{2}Q_m \cdot Q_m, \quad \|Q\|_3 = Q_e \cdot Q_m, \quad (15)$$

where the inner product  $\cdot$  is defined with respect to the defining metric of the non-compact duality group  $G(\mathbb{R})$ . The conjugate complex variables  $\tau_1, \tau_2, \tau_3$  form the Siegel upper half plane  $\mathcal{H}_2$

$$Z = \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_2 \end{pmatrix} \in \mathcal{H}_2. \quad (16)$$

The automorphic groups are Hecke type congruence subgroups  $\Gamma_0^{(2)}(N) \subset \text{Sp}(4, \mathbb{Z})$ , hence the associated forms are Siegel modular forms of genus two.

The key feature of the Siegel modular forms that appear in the context of  $\text{CHL}_N$  black hole entropy is that they are not of general type, but belong to the Maaß Spezialschar, i.e. they are obtained via a combination of the Skoruppa lift [36] from classical modular forms to Jacobi forms, and the Maaß lift [37] from Jacobi forms to Siegel modular forms

$$f(\tau) \in S_{w+2} \xrightarrow{\text{SL}} \varphi_{w,1}(\tau, \rho) \in J_w \xrightarrow{\text{ML}} \Phi_w(\tau, \sigma, \rho) \in S_w, \quad (17)$$

where  $\tau = \tau_1, \sigma = \tau_2, \rho = \tau_3$ . The classical modular form  $f(\tau)$  whose Maaß-Skoruppa lift is the Siegel modular form  $\Phi_w = \text{MS}(f)$  is called the Maaß-Skoruppa root.

### 3.3 Automorphic motives

Given that the entropy of black holes is described by automorphic forms, one can ask whether the spacetime structure of the compactification manifolds leads to motives rich enough to support these automorphic forms. It is not expected that general automorphic forms are of motivic origin, however algebraic automorphic forms are conjectured to be supported by motives. Background material for Siegel forms and motives can be found in [38, 39].

In the special case of Siegel forms of genus two modular forms that appear in the context of  $\text{CHL}_N$  black holes the conjectures concerning the motivic origin indicate that the compactification manifold cannot provide directly the appropriate motivic cycle structure. The easiest way to see this is as follows [3]. Suppose  $M_\Phi$  is a motive whose  $L$ -function  $L(M_\Phi, s)$  agrees with the spinor  $L$ -function  $L_{\text{sp}}(\Phi, s)$  associated to a Siegel modular form  $\Phi$  of arbitrary genus  $g$  and weight  $w$

$$L(M_\Phi, s) = L_{\text{sp}}(\Phi, s). \quad (18)$$

The weight  $\text{wt}(M_\Phi)$  of such genus  $g$  spinor motives follows from the (conjectured) functional equation of the  $L$ -function as

$$\text{wt}(M_\Phi) = gw - \frac{g}{2}(g+1). \quad (19)$$

For the special case of genus 2 spinor motives the Hodge structure takes the form

$$H(M_\Phi) = H^{2w-3,0} \oplus H^{w-1,w-2} \oplus H^{w-2,w-1} \oplus H^{0,2w-3}. \quad (20)$$

It should be noted that this Hodge structure only applies to pure motives. In the case of mixed motives it is possible, for example, that rank 4 motives can give rise to classical modular forms [2].

While the Hodge (20) type of  $M_\Phi$  is that of a Calabi-Yau variety, the precise structure is only correct for modular forms of weight three. It turns out that for the class of  $\text{CHL}_N$  models the weights of the Siegel modular forms take values in a much wider range  $w \in [1, 10]$ . It follows that for most  $\text{CHL}_N$  models the Siegel modular form will take the wrong value to be induced directly by motives in the way usually envisioned in the conjectures of arithmetic geometry.

The same is the case for the classical Maaß-Skoruppa roots, whose weights are given by  $(w+2)$ . The motivic support  $M_f$  for such modular forms  $f$  is of the form

$$H(M_f) = H^{w-1,0} \oplus H^{0,w-1}, \quad (21)$$

hence the only modular forms that can fit into heterotic compactifications have weight two, three, or four.

### 3.4 Lifts of weight two forms

The key to the identification of the motivic origin of the  $\text{CHL}_N$  black hole entropy turns out to be an additional lift construction that interprets the Maaß-Skoruppa roots of weight  $(w+2)$  in terms of modular forms of weight two for all  $N$ , and hence in terms of elliptic curves [3]. These Maaß-Skoruppa roots decompose into two distinct classes of forms, one class admitting complex multiplication, the second class not. For this reason it is not surprising that the lifts of weight two modular forms to the  $\text{CHL}_N$  Maaß-Skoruppa roots involve two different constructions, depending on the type of the higher weight form. For the forms without complex multiplication the lift interpretation of the MS root  $f_{w+2}$  in terms of the weight two form  $f_2 \in S_2$  can be written as

$$f_{w+2}(q) = f_2(q^{1/m})^m, \quad \text{with } m = \frac{1}{2} \left\lceil \frac{24}{N+1} \right\rceil. \quad (22)$$

The lift for the class of Maaß-Skoruppa roots with complex multiplication derives from the existence of algebraic Hecke characters whose  $L$ -functions are the inverse Mellin transform of the MS roots. More details can be found in ref. [3].

The interpretation of the Maaß-Skoruppa roots in terms of weight 2 modular forms  $f_2^{\tilde{N}}$  via these two additional lifts for CM and non-CM forms shows that the motivic origin of the Siegel modular entropy of  $\text{CHL}_N$  models is to be found in elliptic curves. This follows from the fact that for all  $\text{CHL}_N$  models the geometric structure that supports the weight 2 forms is that of elliptic curves  $E_{\tilde{N}}$ , whose conductor  $\tilde{N}$  varies with the order  $N$  of the quotient group  $\mathbb{Z}_N$ . More precisely, the  $L$ -functions associated to both of these objects agree

$$L(f_2^{\tilde{N}}, s) = L(E_{\tilde{N}}, s). \quad (23)$$

Abstractly, this follows from the proof of the Shimura-Taniyama-Weil conjecture [19] and [21], but no such heavy machinery is necessary for the concrete cases based on the  $\text{CHL}_N$  models, where the elliptic curves can be determined explicitly for each  $N$ . This shows that the motivic origin of the Siegel black hole entropy considered in [33, 34, 35] can be reduced to that of lower-dimensional cycles, as opposed to the three-dimensional nature of the compactification manifold  $X_N = T^6/\mathbb{Z}_N$  in the heterotic frame, or  $(K3 \times T^2)/\mathbb{Z}_N$  in the type IIA frame.

## Max

During the course of the work described here I often had reason to think of one of Max's proverbs, which I learned from him. This was in particular the case during the early stages of the program to use motivic modular forms to formulate a new approach to the problem of an emergent spacetime. The relation between CFT-theoretic and motivic modular forms that I was looking for initially remained hidden and it seemed for some time that it actually might not exist. The main problem was that it is a priori not clear which, if any, conformal field theoretic modular forms on the worldsheet might be useful as building blocks of the motivic  $L$ -series of the compact dimensions. This was cause for some aggravation, and while thinking about this I often recalled one of Max's favorite comments, who in such times reminded himself and his friends of the proverb "Mühsam ernährt sich das Eichhörnchen". It is an interesting open problem to find a translation that properly conveys the meaning of this important insight into human nature.

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